

Quantum Group Approach to a soluble vertex model with generalized ice-rule

L. SOW CIRE^{†*} and T. T. TRUONG^{‡†}

[†]Laboratoire de Modèles de Physique Mathématique,
Département de Physique, Université de Tours,
Parc de Grandmont, F-37200 Tours, France.

[‡]Groupe de Physique Statistique,
Département de Physique, Université de Cergy-Pontoise,
B.P 8428 F-95806 Cergy-Pontoise Cedex-France.

(to appear in Int. J. Mod. Phys. A (1995))
PACS Numbers: 05.70.JK, 64.60.Fr, 75.40.-s.

Abstract

Using the representation of the quantum group $SL_q(2)$ by the Weyl operators of the canonical commutation relations in quantum mechanics, we construct and solve a new vertex model on a square lattice. Random variables on horizontal bonds are Ising variables, and those on the vertical bonds take half positive integer values. The vertices is subjected to a generalized form of the so-called “ice-rule”, its property are studied in details and its free energy calculated with the method of quantum inverse scattering. Remarkably in analogy with the usual six-vertex model, there exists a “Free-Fermion” limit with a novel rich operator structure. The existing algebraic structure suggests a possible connection with a lattice neutral plasma of charges, via the Fermion-Boson correspondence.

*e-mail: sowcire@celfi.phys.univ-tours.fr

†e-mail: truong@u-cergy.fr

Introduction

Vertex systems which are originally introduced to two dimensional ferroelectrics models with the so called “ice-rule”, have rapidly evolved in the last quarter of century and have led to a tremendous development in the statistical physics of soluble models. New concepts and new mathematical structures have been discovered essentially due to the introduction of the method of quantum inverse scattering. Recently the solubility conditions known as the Yang-Baxter equations, which generalized the star-triangle relations for spin systems, for vertex systems are shown to have an expression in terms of quantum groups. These new objects introduced by Drinfeld [1] and Jimbo [2], are essentially matrix groups with non commutative elements. In this paper we shall be concerned with the $SL_q(2)$ group which is a q -deformation of the standard $SL(2)$ group [3].

Interestingly, when one seeks a representation of the $SL_q(2)$ group by the Weyl operators associated with a quantum degree of freedom Q and its canonical conjugate momentum P , one obtains a new soluble vertex system having on horizontal bonds random variables taking only two values ($\pm 1/2$), and on vertical bonds random variables taking an infinite number of values (a kind of infinite spin). Physically one may view the system as a two-dimensional array of vertical quantized spring coupled to binary horizontal devices. Alternatively the discrete vertical variables are analogous to “heights” in face models (or SOS models) of statistical mechanics for which, there exists a Yang-Baxter Algebra and Bethe-Ansatz solutions [4], whereas the horizontal variables remain standard arrow variables of Lieb’s six-vertex model. In this respect, the new soluble model may be viewed as having a mixed face-vertex nature.

Section 1 is devoted to the description of the system. We show how the construction of the vertex operator L is performed using Q and P . We also indicated how a local vacuum may be chosen in order to be able to apply the method of quantum inverse scattering and to obtain the Bethe Ansatz equations.

In section 2, we diagonalized the transfer matrix of the model, using this conventional technique and obtain the free energy per site. In analogy to the six-vertex model, we derive the tensor representation of the $SL_q(2)$ group, the generators of which have a remarkable structure which is parallel to that of the six-vertex model. At the value $q = i$, there is also a “free fermion” limit, and fermion-like operators may be expressed in terms of the Bose degrees of freedom Q_j and P_j for $j = 1, 2, \dots, N$. The generators of the tensor representation of $SL_q(2)$ in this model, are closed to the “screening” operators of Dotsenko and Fateev [5] in their treatment of Conformal Field theory using the Coulomb gas picture. If a Hamiltonian operator for a chain with degrees of freedom Q_j and P_j can be found so that it commutes with the transfer matrix of the model, one

may be able to establish a connection with a lattice Coulomb gas. Since the critical behavior of these models are the same, there is a strong evidence that such a connection may exist. We conclude by comparing our model with those arising from the lattice version of the quantum Sine-Gordon or the quantum Non-linear Schrödinger equation (Faddeev , Korepin, Kulish, Sklyanin et al. [6], [7], [8], and those arising from the bosonisation of the six-vertex model using the Holstein-Primakoff transformation (Y. K. Zhou [9]). Finally we give some future directions of investigation.

1 The vertex system on a square lattice

1.1 Formulation

The statistical system we consider is made up of elementary vertices consisting of a pair of Ising variables σ and σ' on horizontal bonds and other random variables on vertical bonds ξ and ξ' . For each set of values of the 4 random variables σ , σ' and ξ , ξ' a Boltzmann weight $W(\xi, \xi'; \sigma, \sigma')$ is assigned (see fig.1).

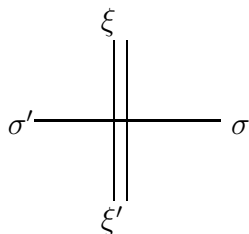


Figure 1: The elementary vertex with Boltzmann weight $W(\xi, \xi'; \sigma, \sigma')$.

In the 1970's, R.J.Baxter [10] showed that vertices that satisfy the triangle-relations (nowadays called the Yang-Baxter Equations) then their horizontal row transfer matrices form a commuting set of operators with respect to a "spectral parameter" introduced by the Russian school [6]. The triangle equations are the analog of the star-triangle relations for spin systems which yield then the same property of commuting horizontal row transfer matrices. The triangle-relations state that the partition function of the following two triangles are the same for every configuration of random variables on open external bonds (i.e. $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \xi_1, \xi_2$) (see fig.2).

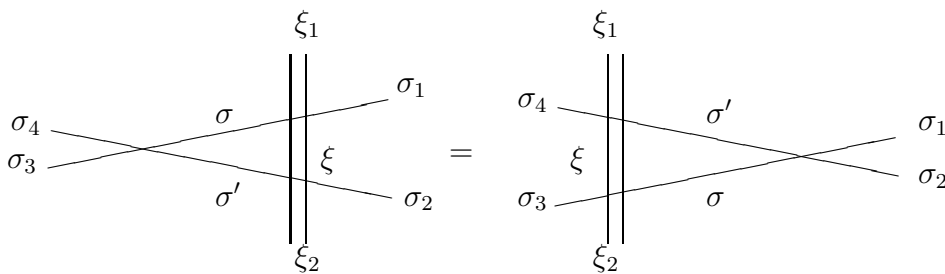


Figure 2: Summation is performed on σ , σ' and, ξ

Here as it stands, we note the necessity of having a third vertex having only Ising variables on the left and right sides. The triangle-relations are the necessary conditions for the calculation of the partition function of the model by Bethe Ansatz techniques and consequently the thermodynamics of the system. Note that when $\xi = \sigma = \pm 1$, we recover the standard triangle relations of the six or eight vertex solved by E. H. Lieb [11] and R. J. Baxter [10]. A particular system of vertex having unequal number of random variables on horizontal and vertical bonds but verifying the triangle relations was solved by R. Z. Bariev and Yu. V. Kozhinov [12], and a general discussion on such type of vertex is presented by H. J. de Vega [13]

An appropriate way of handling the star triangle relations (or Yang-Baxter equations) consists of using an operator formulation. We associate to a vertex with Ising variables on horizontal bonds a 2×2 matrix whose matrix elements $L_{\sigma\sigma'}$ are labelled by σ and σ' (see fig.1):

$$L = \begin{pmatrix} L_{11} & = & \alpha & & L_{1-1} & = & \beta^- \\ L_{-11} & = & \beta^+ & & L_{-1-1} & = & \delta \end{pmatrix}$$

$\alpha, \beta^\pm, \delta$ are themselves operators in a “vertical” Hilbert space with matrix elements labelled by ξ and ξ' . Anticipating on the existence of a “spectral parameter” [14], the triangle-relations take up the following compact form:

$$R\left(\frac{u}{v}\right)L(u) \otimes L(v) = L(v) \otimes L(u)R\left(\frac{u}{v}\right), \quad (1)$$

where R is associated with the standard vertex with only Ising variables on the bonds.

Eq(1) is rather general, one may consider the set of all L satisfying (1) with the same R ; in particular one may choose R to be that of a standard symmetric six-vertex model. In this case the matrix elements $\alpha(u), \delta(u), \beta^\pm$ of L are shown to be expressible in terms of the generators of the quantum group $SL_q(2)$: L_z, L^+, L^- which obey the defining relations (see [15]):

$$\begin{aligned} [L_z, L^\pm] &= \pm L^\pm \quad \text{or} \quad q^{L_z} L^\pm = q^{\pm 1} L^\pm q^{L_z}, \\ [L^+, L^-] &= \frac{q^{2L_z} - q^{-2L_z}}{q - q^{-1}}. \end{aligned} \quad (2)$$

Then one has the expressions (see Wiegmann and Zabrodin [15])

$$\begin{aligned} \beta^\pm &= L^\pm, \\ \alpha(u) &= \frac{uq^{L_z} - u^{-1}q^{-L_z}}{q - q^{-1}}, \end{aligned} \quad (3)$$

$$\delta(u) = \frac{uq^{-L_z} - u^{-1}q^{L_z}}{q - q^{-1}}. \quad (4)$$

Thus the problem is reduced to finding the appropriate representations of $SL_q(2)$ which corresponds to the definition of the vertex. We observe that the standard six vertex model is recovered if one considers the spin 1/2 representation of $SL_q(2)$ which is generated by the Pauli matrices, higher spin representations are possible (see Saleur and Pasquier [16], [17]).

But elaborating on Wiegmann and Zabrodin [15] representation of $SL_q(2)$ by 2-dimensional magnetic translation operators, we shall consider next, the representation of $SL_q(2)$ by the Weyl operators of the canonical commutation relation of one degree of freedom in quantum mechanics.

1.2 Algebraic tools

In his treatement of quantum mechanics H. Weyl had proposed to replace the canonical commutator between a dynamical variable Q and its conjugate P

$$[Q, P] = iI, \quad (5)$$

by the commutation relation between e^{ipQ} and e^{ixP} :

$$e^{ixP}e^{ipQ} = e^{ixp}e^{ipQ}e^{ixP}. \quad (6)$$

In this paper we shall concentrate on the case $q = e^{i\eta}$, which corresponds to a physical phase of the system. Using appropriate scaling, (6) may be rewritten under the form:

$$e^{iP}q^Q = qq^Qe^{iP}. \quad (7)$$

This relation allows us to represent the generators of $SL_q(2)$ as:

$$\begin{aligned} L_z &= Q, \\ L^+ &= \frac{q^{Q-1/2} - q^{-Q+1/2}}{q - q^{-1}} e^{-iP}, \\ L^- &= -e^{iP} \frac{q^{Q-1/2} - q^{-Q+1/2}}{q - q^{-1}}. \end{aligned} \quad (8)$$

From (8), we recover in the limit $q \rightarrow 1$, the following representation of $SL(2)$ generators:

$$\begin{aligned} S^+ &= (Q - 1/2) \exp(-iP), \\ S^- &= -\exp(iP)(Q - 1/2), \\ S^z &= Q. \end{aligned}$$

We observe that L^+ and L^- are each other antihermitian when we require that the limit $q \rightarrow 1$ of (8) obeys the commutators of $SL(2)$.

One may check that (8) fulfills automatically (2), using the shift property of e^{ixP} ,

$$e^{ixP} Q e^{-ixP} = Q + x. \quad (9)$$

The new representation of the vertex operator L is now:

$$\begin{aligned} \beta^+ &= \frac{q^{Q-1/2} - q^{-Q+1/2}}{q - q^{-1}} e^{-iP}, & \beta^- &= -e^{iP} \frac{q^{Q-1/2} - q^{-Q+1/2}}{q - q^{-1}}, \\ \alpha &= \frac{uq^Q - u^{-1}q^{-Q}}{q - q^{-1}}, & \delta &= \frac{uq^{-Q} - u^{-1}q^Q}{q - q^{-1}}. \end{aligned} \quad (10)$$

Physically we have a quantum mechanical degree of freedom on “vertical” space coupled to Ising spins on horizontal bonds. Such a system obeys the triangle-relations (Yang-Baxter equations) and may under specified conditions be solved by Bethe Ansatz techniques. Since it has the R -matrix of a six-vertex model, one expects its critical behavior to be the same as in the six-vertex case. The attractive point is that the critical universality class may be in fact defined by the choice of the R -matrix.

For comparison let us recall the standard spin 1/2 representation of $SL_q(2)$, for which we have:

$$\frac{q^{\sigma^z} - q^{-\sigma^z}}{q - q^{-1}} = \sigma^z.$$

In this case with the standard Pauli matrices σ^x , σ^y , σ^z , one has:

$$\begin{aligned} \beta^\pm &= \sigma^\pm = 1/2(\sigma^x \pm i\sigma^y), \\ \alpha &= \frac{uq^{\sigma^z/2} - u^{-1}q^{-\sigma^z/2}}{q - q^{-1}}, \end{aligned}$$

$$\delta = \frac{uq^{-\sigma^z/2} - u^{-1}q^{\sigma^z/2}}{q - q^{-1}}. \quad (11)$$

Here the vertical space is two-dimensional, whereas it is infinite dimensional when one uses P and Q .

1.3 Schrödinger representation and the local vacuum.

In order to apply the method of quantum inverse scattering (Faddeev [14]) to construct the explicit solution of the problem, one needs to construct a local vacuum. This is fairly evident in the case of the standard six-vertex model where one may choose for example, the state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which is annihilated by the $\beta^- = \sigma^-$ operator:

$$\sigma^- \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0. \quad (12)$$

Thus the local vacuum is nothing else as the “spin down” state on the vertical direction. Moreover

$$\begin{aligned} \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \frac{uq^{-1/2} - u^{-1}q^{1/2}}{q - q^{-1}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \delta \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \frac{uq^{1/2} - u^{-1}q^{-1/2}}{q - q^{-1}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \quad (13)$$

Or in a more usual parametrization : $q = e^{i\eta}$, $u = e^\theta$; we recognize:

$$a = \frac{\sinh(\theta - i\eta/2)}{\sinh i\eta}, \quad b = \frac{\sinh(\theta + i\eta/2)}{\sinh i\eta}.$$

With the use of Q and P it is necessary to find a local vacuum. We shall do so in using the position Schrödinger representation, in which the Q operator is diagonal and has continuous spectrum:

$$\begin{aligned} Q|\xi\rangle &= \xi|\xi\rangle, \\ e^{ixP}|\xi\rangle &= |\xi - x\rangle. \end{aligned} \quad (14)$$

The local vacuum $|\omega\rangle$, in analogy with eq(12), is defined by the annihilation property

$$\begin{aligned} \beta^-|\omega\rangle &= L^-|\omega\rangle = 0 \\ e^{iP} \frac{q^{\omega-1/2} - q^{-\omega+1/2}}{q - q^{-1}} |\omega\rangle &= 0. \end{aligned} \quad (15)$$

Hence one must choose

$$\omega = 1/2. \quad (16)$$

The existence of this local vacuum is directly related to the construction of the representation (see eq.(8)) with a proper limit $q \rightarrow 1$. Demanding from the start that L^+ and L^- be each other hermitian, will not lead to the correct $q \rightarrow 1$ limit, nor yield a local vacuum for one vertex operator L , as in the Sine-Gordon theory [8]. Application of $\beta^+ = L^+$ on the local vacuum $|\omega\rangle = |1/2\rangle$ yields

$$L^+|1/2\rangle = |3/2\rangle.$$

More generally we have:

$$(L^+)^n|1/2\rangle = \frac{q^n - q^{-n}}{q - q^{-1}} \times \frac{q^{n-1} - q^{-n+1}}{q - q^{-1}} \times \dots \times \frac{q - q^{-1}}{q - q^{-1}} |n + 1/2\rangle. \quad (17)$$

The sequence of states $|n + 1/2\rangle$ reminds us of the sequence of the harmonic oscillator with unit frequency. In this sense it is reasonable to think of the vertices as vertical springs coupled to horizontal Ising variables.

We may evaluate also, since Q is diagonal, the action of $\alpha(u)$ and $\delta(u)$ on $|1/2\rangle$

$$\begin{aligned} \alpha(u)|1/2\rangle &= \frac{uq^{1/2} - u^{-1}q^{-1/2}}{q - q^{-1}} |1/2\rangle, \\ \delta(u)|1/2\rangle &= \frac{uq^{-1/2} - u^{-1}q^{1/2}}{q - q^{-1}} |1/2\rangle. \end{aligned} \quad (18)$$

This will be used later in constructing the solution.

We can check that the set of states $|n + 1/2\rangle$ generated by application of $(L^+)^n$ on $|1/2\rangle$, the local vacuum, forms an orthonormal set of states for the “vertical” Hilbert space. The proof is as followed: from (2) we have,

$$[L^-, (L^+)^m] = -m(L^+)^{m-1} \frac{q^{2(Q+m-1)} - q^{-2(Q+m-1)}}{q - q^{-1}}. \quad (19)$$

Thus

$$\langle n + 1/2 | m + 1/2 \rangle = \langle 1/2 | (L^-)^n (L^+)^m | 1/2 \rangle.$$

Using the former equation, we arrive at:

$$\langle n + 1/2 | m + 1/2 \rangle = \prod_{j=m-n+1}^m s_j \langle 1/2 | (L^-)^{m-n} | 1/2 \rangle,$$

where

$$s_j = -j \frac{q^{(1/2+j-1)} - q^{-(1/2+j-1)}}{q - q^{-1}}.$$

This is the reason why the model is closed to the six-vertex and may be considered as an extension of the “anisotropic” vertices considered by Bariev [12], where one of the “vertical” random variable may take infinite number of values. The vertical operators $\alpha, \delta, \beta^\pm$ may be then represented in this base by infinite dimensional matrices and one may consider discussing the rotated representation (see Vega) [13].

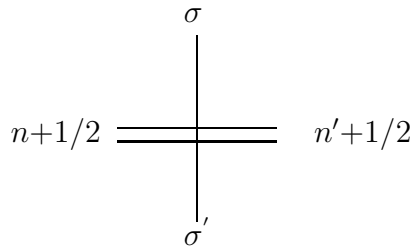


Figure 5: The rotated representation of the vertices.

Because of the “ice-rule”, the weights $\omega_{\frac{1}{2}-\frac{1}{2}}$ and $\omega_{-\frac{1}{2}\frac{1}{2}}$ will appear always in pairs. We may then take out a factor i and redefine the “spectral parameter” as $u' = -iu$, with u' real, to obtain:

$$\begin{aligned} \omega_{\frac{1}{2}\frac{1}{2}} &= \frac{u'q^{n+1/2} + (u')^{-1}q^{-(n+1/2)}}{q - q^{-1}} & \omega_{\frac{1}{2}-\frac{1}{2}} &= i \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}}, \\ \omega_{-\frac{1}{2}\frac{1}{2}} &= -i \frac{q^n - q^{-n}}{q - q^{-1}} & \omega_{-\frac{1}{2}-\frac{1}{2}} &= \frac{u'q^{-(n+1/2)} + (u')^{-1}q^{n+1/2}}{q - q^{-1}}. \end{aligned} \quad (21)$$

Then the partition function will appear as positive definite. This unphysical form of weights arises often in the derivation of the parametrisation of the weights using the Yang-Baxter equations (see for example the case of the eight-vertex model [18]).

Finally we observe that the Casimir operator of the $SL_q(2)$ representation is independent of q , and has the value 0. This may be checked by direct calculation (see [15]):

$$C = \left(\frac{q^{-1/2}q^Q - q^{1/2}q^{-Q}}{q - q^{-1}} \right)^2 + \beta^+ \beta^- = 0.$$

2 Properties of the vertex system

2.1 Quantum inverse scattering and diagonalisation of the transfer matrix

Following standard procedure we construct the monodromy operator $T(u)$, for one row of N vertices (see fig.5).

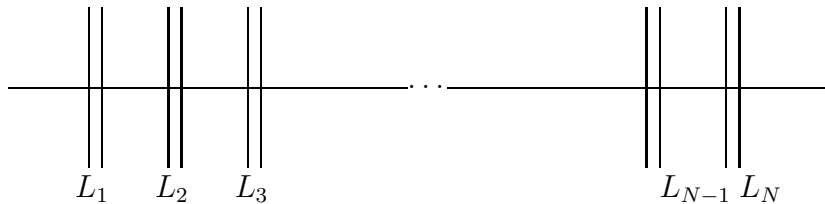


Figure 6: A row of vertices.

As it is known, $T(u)$ is a 2×2 matrix of the type,

$$T(u) = \begin{pmatrix} A(u) & C(u) \\ B(u) & D(u) \end{pmatrix} \quad (22)$$

and obeys the triangle-equations (Yang-Baxter equations) which yield commutation relations between $A(u)$, $B(u)$, $C(u)$ and $D(u)$, necessary to construct the eigenstates of the transfer matrix

$$t(u) = \text{Tr } T(u) = A(u) + D(u). \quad (23)$$

But before doing so, let us recall that due to our choice of local vacuum $|1/2\rangle$ at each site, the local vertex system fulfills an extended “ice-rule” on which, conservation laws rely on. The bare vacuum of the row is thus:

$$|\Omega\rangle = |1/2\rangle_1 \otimes |1/2\rangle_2 \otimes \dots \otimes |1/2\rangle_N. \quad (24)$$

Excitations are created in a standard way by repeated applications of the $B(u)$ operator on $|\Omega\rangle$, namely:

$$B(u_1)B(u_2)\dots B(u_m)|\Omega\rangle = |u_1, u_2, \dots, u_m\rangle. \quad (25)$$

The requirement that $|u_1, u_2, \dots, u_m\rangle$ should be eigenstate of $t(u)$ forces the $u_j, j = 1, 2, \dots, m$, to obey a system of nonlinear equations, the so-called Bethe Ansatz equations:

$$\frac{\sinh(\theta_j + i\frac{\eta}{2})}{\sinh(\theta_j - i\frac{\eta}{2})} = - \prod_{k=1}^m \frac{\sinh(\theta_j - \theta_k + i\eta)}{\sinh(\theta_j - \theta_k - i\eta)}, \quad (26)$$

where $u_j = \exp \theta_j$.

These equations may be solved exactly in the limit of $N \rightarrow \infty$ by standard Fourier techniques since they are formally the same equations as in the six-vertex model [13]. The free energy per site is then in the thermodynamics limit in the antiferroelectric regime for example,[13]

$$f(\theta, \eta) \approx \int_0^{+\infty} \frac{\sinh 2x\theta \sinh [x(\pi - \eta)]}{\cosh(x\eta) \sinh(x\pi)} dx + \text{const}, \quad (27)$$

and has the same critical behavior as the six-vertex or model I of [4]. This is not new since the lattice version of the Sine-Gordon model (Faddeev et al. [14], Korepin et al. [8]) shares also the same behavior. One may say that so long one has the same R -matrix in the Yang-Baxter equations, one necessarily obtain the same critical behavior and conjecture that, a possible classification of universality classes of critical behavior, may be made according to a classification of R -matrices.

2.2 Alternative Quantum Group structure and representation

In this section we discuss some aspects of $SL_q(2)$ which are relevant for the so called “Free Fermion” limit: $q \longrightarrow i$ (or $\eta \longrightarrow \frac{\pi}{2}$), which is well known in the Bethe Ansatz equations (26) (see for example [21]).

Consider the representation of one vertex operator L in term of the generators L_z, L^\pm of $SL_q(2)$. We may form two new operators a^+ and a^- by defining:

$$a^\pm = q^{L_z} L^\pm. \quad (28)$$

Then $SL_q(2)$ may be characterized by L_z, a^\pm with the following relations:

$$\begin{aligned} q^{L_z} a^\pm &= q^{\pm 1} a^\pm q^{L_z}, \\ a^+ a^- - q^{-2} a^- a^+ &= \frac{q^{4L_z} - 1}{q^2 - 1}. \end{aligned} \quad (29)$$

Curiously this last equation appears as a q -commutator for a q -deformed oscillator, although not identical to the relation proposed by Biedenharn and Macfarlane [3]. The limit $q \longrightarrow 1$ of (29) is again $SL(2)$ and the oscillator algebra can be recovered after a Wigner contraction is performed [19].

Moreover we see that for any integer m ,

$$(a^+)^m = q^{\frac{-m(m-1)}{2}} q^{mL_z} (L^+)^m. \quad (30)$$

There exists a similar equation for $(a^-)^m$. Note that a^+ and a^- are not each other hermitian.

As it is known [13], one may obtain tensor representation of $SL_q(2)$ by repeated application of the coproduct ([2]) on the generators for one vertex. This is what one obtains physically when the whole row of N vertices is considered. The N -fold tensor representation generators are:

$$J_\pm = \sum_{j=1}^N q^{(L_z)_1} \otimes \dots \otimes q^{(L_z)_{j-1}} \otimes L_j^\pm \otimes q^{-(L_z)_{j+1}} \otimes \dots \otimes q^{-(L_z)_N},$$

$$J_z = \sum_{j=1}^N (L_z)_j, \quad (31)$$

they fulfill the defining relations (2):

$$[J_+, J_-] = \frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}},$$

$$q^{J_z} J_{\pm} = q^{\pm 1} J_{\pm} q^{J_z}. \quad (32)$$

These relations may be obtained by hand starting from the standard commutation relations of $A(u)$, $B(u)$, $C(u)$ and $D(u)$, obtained in the quantum inverse scattering method, in the ferroelectric regime (q real and positive), and taking the appropriate limits [13].

Similarly the tensor representation may be characterized by the set of operators J_z , η^{\pm} where,

$$\eta^{\pm} = q^{J_z} J_{\pm} = \sum_{j=1}^N q^{2(L_z)_1 + \dots + 2(L_z)_{j-1}} a_j^{\pm}. \quad (33)$$

Of course η^{\pm} , J_z obey the relation (29) satisfied by a^{\pm} , L_z . But here, we have the freedom of introducing new local operators η_j^{\pm} defined by:

$$\eta_j^{\pm} = q^{2(L_z)_1 + \dots + 2(L_z)_{j-1}} a_j^{\pm}, \quad (34)$$

which in turns obey the following commutation relations:

$$\begin{aligned} \eta_j^+ \eta_i^+ &= q^{2\text{sgn}(j-i)} \eta_i^+ \eta_j^+, \\ \eta_j^- \eta_i^- &= q^{-2\text{sgn}(j-i)} \eta_i^- \eta_j^-, \\ \eta_j^+ \eta_i^- &= q^{-2} \eta_i^- \eta_j^+ \quad i \neq j, \end{aligned}$$

and

$$\eta_j^+ \eta_j^- - q^{-2} \eta_j^- \eta_j^+ = q^{4(L_z)_1 + \dots + 4(L_z)_{j-1}} \frac{q^{4(L_z)_j} - 1}{q^2 - 1}. \quad (35)$$

Note that $(\eta_j^+)^m$ or $(\eta_j^-)^m$ for m integer, is not necessarily zero, because of (30).

Consequently in this section, we have shown that the structure of $SL_q(2)$, presents aspects of the so called q -deformed oscillator of a special type and that, its tensor representation leads naturally to local operators which exhibit “any-
onic” commutation relations as well as behaves the q -deformed oscillator [3], [20].

2.3 Free Fermion limit

As can be seen from eq (26), the Bethe Ansatz equations decouple at $\eta = \frac{\pi}{2}$ (or $q = i$): the scattering of pseudo-particles created by the $B(u)$ operators becomes trivial, the corresponding phase shifts equal -1, and the multiparticle wave function is simply represented by a Slater determinant.

We can now make some interesting observations in two different situations: the six-vertex case and the vertex studied in this paper.

2.3.1 The six-vertex case

The η^\pm operators are closely related to usual lattice Fermion operators. Since $L_z = \frac{1}{2}\sigma^z$,

$$\eta_j^\pm = e^{i\frac{\pi}{2}(\sigma_1^z + \dots + \sigma_{j-1}^z)} e^{i\frac{\pi}{4}\sigma_j^z} \sigma_j^\pm, \quad (36)$$

and they anticommute:

$$\begin{aligned} \eta_j^\pm \eta_i^\pm &= -\eta_i^\pm \eta_j^\pm, \\ \{\eta_j^+, \eta_i^-\} &= \delta_{ij} e^{i\pi(\sigma_1^z + \dots + \sigma_{j-1}^z)} \frac{e^{i\pi\sigma_j^z} - 1}{-2} = \delta_{ij} (-1)^{j-1}. \end{aligned} \quad (37)$$

Moreover $(\eta_j^\pm)^2 = 0$. Thus we may define Fermion operators by:

$$c_j^\pm = (i)^{j-1} \eta_j^\pm = \exp\left\{i\frac{\pi}{2}\left(\sum_{l=1}^{j-1}(\sigma_l^z + 1)\right)\right\} \exp\left\{i\frac{\pi}{4}\sigma_j^z\right\} \sigma_j^\pm, \quad (38)$$

which are not exactly identical to the ones obtained by the Jordan-Wigner transformation, since c_j^\pm are not hermitian adjoint each other [11], but they anticommute and are of square zero.

2.3.2 The new vertex case

The η_j^\pm operators do have a fermion-like behavior. Here $L_z = Q$,

$$\eta_j^\pm = e^{i\pi(Q_1 + \dots + Q_{j-1})} e^{i\frac{\pi}{2}Q_j} \beta_j^\pm. \quad (39)$$

They also anticommute:

$$\begin{aligned} \eta_j^\pm \eta_i^\pm &= -\eta_i^\pm \eta_j^\pm, \\ \{\eta_j^+, \eta_i^-\} &= \delta_{ij} e^{2i\pi(Q_1 + \dots + Q_{j-1})} \frac{e^{2i\pi Q_j} - 1}{-2}. \end{aligned} \quad (40)$$

Now the r.h.s of the second equation of (40) must be applied to the N-fold tensor product states of the type

$$\otimes_{l=1}^N |n_l + 1/2\rangle \quad n_l = 0, 1, 2, \dots \quad (41)$$

Since all Q_j are diagonal in this representation, we obtain a factor $(-1)^{j-1}$ as in the second equation of (37):

$$e^{2i\pi\{(n_1+\dots+n_{j-1})+\frac{(j-1)}{2}\}} \frac{e^{2i\pi(n_j+\frac{1}{2})} - 1}{-2} = (-1)^{j-1}. \quad (42)$$

Hence in analogy with the six-vertex case, we may define the fermion-like operators:

$$d_j^\pm = (i)^{j-1} \eta_j^\pm, \quad (43)$$

which anticommute properly as c_j^\pm in (37), but its square (or any power of it), is not zero because:

$$(a_j^+)^m = e^{-i\frac{\pi}{4}m(m-1)} e^{i\frac{\pi}{2}mQ_j} (\beta_j^+)^m, \quad (44)$$

is not zero at site j . This feature already manifests itself in the Bethe-Ansatz wave function which differs markedly with the Bethe-Ansatz wave function in the six-vertex model. However since the fermionic character prevails at the point $q = i$ ($\eta = \frac{\pi}{2}$), there is a simplifying decoupling in the Bethe-Ansatz equations and perhaps, a simple direct evaluation of the free energy is possible. In any case, this study suggests that the old duality in one dimension between Fermions and Bosons discovered in the continuum, has an unexpected features on the lattice which are brought to light by the algebraic structure of quantum groups.

3 Conclusion and outlook

We have presented here a study of an integrable vertex system made of Ising variables on horizontal bonds and, “oscillator-like” variables on vertical bonds. The model is integrable in the sense it fulfills the Yang-Baxter equations with an R -matrix identical to the six-vertex model one. This fact leads to the same critical behavior of the free energy per site as in the six vertex case [21]. The detailed algebraic structure is however different. The Bethe Ansatz wave function is not in general antisymmetric and the free fermion limit reveals a fermionic behavior of the creation/annihilation operators for the excitations.

In the past, there has been several studies of this type of the vertex. Faddeev et al., Korepin et al. [6], [8], [14], in the Sine-Gordon model on a lattice (or its non relativistic limit the non-linear Schrödinger model), have considered an integrability based on two sites in order to achieve solubility of the model. Zhou [9] has performed the Holstein-Primakoff transformation on the six-vertex model. Here we have directly obtained the “vertical Bose variables” by the technique of quantum groups, and have obtained an integrability based only on one site.

The present study opens up new lines of investigation. First it would be interesting to find the Hamiltonian of the system, since the Hamiltonian of the

six-vertex model is the XXZ chain. Presumably, such a Hamiltonian under appropriate continuum limit may yield the Sine-Gordon Hamiltonian. Second, the limit of free Fermion may provide also new insight in the old equivalence between Fermion and Boson in one dimension. Finally one may choose formal perturbation theory to generate, as it is known in the continuum, a classical Coulomb gaz of “electric charges” in the plane. But on a lattice, the spacing between sites would then be a parameter controlling the divergences which one has to introduce by hand in a standard bosonisation procedure. We hope to tackle these problems in the future.

Acknowledgment

One of us (S.C.L) would like to thank the “Conseil Régional de la Région Centre” for financial support.

We thank the referee for his comments leading to the improvement of the paper.

References

- [1] V. G. Drinfeld, *Dokl. Akad. Nauk*, **283**, 1060 (1985).
- [2] M. Jimbo, *Lett. Math. Phys.*, **10**, 63 (1985).
- [3] A. J. Macfarlane, *J. Phys.*, **A22**, 4381 (1989),
L. C. Biedenharn, *J. Phys.*, **A22**, L873 (1989).
- [4] H. J. de Vega, *Int. J. mod. Phys.*, **A5**, 1611 (1990).
- [5] V. S. Dotsenko and V. A. Fateev, *Nucl. Phys.*, **B240** (1984).
E. H. Lieb, *Phys. Rev. Lett.*, **18**, 1046 (1967).
- [6] E. K. Sklyanin, L. A. Takhtadzhyan & L. D. Faddeev, *Theor. Math. phys.*, **40**, 698 (1979).
- [7] P.P. Kulish & E. K. Sklyanin, Tvärmine lectures, Springer Lecture Notes in physics, **151** (1981).
- [8] A. G. Izergin & V. E. Korepin, *Lett. Math. Phys.*, **5**, 199 (1981); *Nucl. Phys.*, **B205**, 401 (1982).
- [9] Y. K. Zhou, *Int. J. Mod. Phys.*, **A7**, 6385 (1992).
- [10] R. J. Baxter, *Ann. Phys. (N. Y.)*, **70**, 193 (1972).
- [11] E. H. Lieb, T. Schultz & D. Mattis, *Ann. Phys. (N. Y.)*, **16**, 407 (1961).

- [12] R. Z. Bariev & Yu. V. Kozhinov, *Physics Lett.*, **92A**, 79 (1982).
- [13] H. J. de Vega, *Adv. Stud. Pure Math.*, **19**, 567 (1989); *Int. J. Mod. Phys.*, **4**, 2371 (1989).
- [14] L. D. Faddeev, *Sov. Sci. Review*, **C1**, 107 (1980); *Int. J. Mod. Phys.*, **A10**, 1845 (1995).
- [15] P. B. Wiegmann & A. V. Zabrodin *Nucl. Phys.*, **B422**, 495 (1994).
P.P. Kulish & E.K. Sklyanin, *J. Phys.*, **A24**, L435 (1991).
- [16] H. Saleur, in *Recent Developments in Conformal Field Theories* Trieste October 1989, Saclay Preprint SPHT/89-186.
- [17] V. Pasquier H. Saleur, *Nucl. Phys.* **B230**, 523 (1990).
- [18] R. J. Baxter, *Phys. Rev. Lett.* **26**, 832 (1971).
- [19] Robert Gilmore, *Lie Groups, Lie Algebras and their applications* (Wiley-Interscience, 1974).
- [20] F. Wilczek, *Phys. Rev. Lett.*, **48**, 1144 (1982), *ibid* **49**, 957 (1982),
J. M. Leinaas, J. Myrheim, *Nuovo Cimento*, **37B**, 1 (1977).
- [21] R. J. Baxter, *Exactly solved models in statistical mechanics* (Academic Press, N. Y., 1982).